

Squeezing: can we extend the algorithm?

B. Mielnik and J. Fuentes

Departamento de Física, CINVESTAV-IPN.

Abstract

In the traditional studies of the squeezing, the attention is focused mostly on the particular squeezed states and their evolution, less on the dynamical operations which could squeeze simultaneously some wider families of quantum states, independently of their initial shape. The paper below looks for new steps in this direction, achieved by softly acting electromagnetic fields which might produce the squeezing of the canonical q, p observables of charged particles. If real, it may turn of interest for quantum tomography, as well as for the critical analysis of some fundamental quantum problems. It is also noticed that the most elementary case of the Toeplitz algebra provides much easier solutions of the operational problems than the recently applied Ermakov-Milne invariants.

I Introduction

IN SOME BASIC control problems two pictures of quantum theory are competing: that of Schrödinger (state evolution) and of Heisenberg (the evolution of observables). The advantages of Heisenberg's representation were strongly defended in a polemic article of Dirac, "*Quantum electrodynamics without the dead wood*" [1]. Even if one does not share his criticism, some of its practical aspects revive in the quantum control problems. It concerns specially the linear transformations of the canonical variables q, p (including the squeezing) described in terms of operator algebras [2]. One of examples was the discussion about the possibility of an exact and fast measurement of the free particle position. For some time it was believed that such measurement should obey the *standard quantum limit* (SQL). In 1983 Yuen showed that SQL can be defeated by elementary arguments using the *twisted* canonical observables [3]. By looking carefully, his ideas were the first glimpse into the future *quantum tomography* [4, 5, 6] describing the quantum states in terms of "tomographic images" defined via the Radon transforms [7, 8] of the canonical operators. Some consequences of this view for the basic QM ideas, though not immediate, are not negligible.

In the formalism of quantum mechanics every self-adjoint operator in the Hilbert space is an *observable* representing a measuring device which performs an instantaneous check, "reducing" the wave packet. Yet, the laboratory practice suggests less abstract images in which the act of measurement is prepared by a previous linear (unitary) evolution and the fundamental (nondeterministic?) choice occurs at the very end. Some controversies still exist, *i.e.*, whether the decoherence is sufficient to explain the effect (see an old but still valid objection of Bohm and Bub [9], illustrated perhaps by the highly unstable ion behavior, described by S. Haroche [10]). Note also the doubts whether the observation "reduces" only the quantum state in the very instant of measurement, or reduces also the previous evolution history (Wheeler [11]). It turns out that the ideas of the dynamical squeezing are not independent of these unsolved questions.

Below, we discuss the squeezing effects for charged particles in a class of time dependent Hamiltonians. We consider the variable external fields as the only credible source of the phenomenon. So, we skip all formal results obtained for time dependent masses, material constants, etc. In order to avoid complications of radiative effects, we focus attention on slow evolution processes. For convenience, our mathematical calculations are carried in dimensionless variables but the results are then translated into the physical units.

If we are so much interested in generating the pure quantum states (represented by vectors in a Hilbert space) as well as the dynamical evolution operations (represented by the unitary operators) it is due to the assumed universality of the above mathematical language, which is supposed to be at the bottom of all quantum mechanical problems. However, is it indeed universal? Can indeed all vectors (rays) in a complex Hilbert space correspond to some real quantum states? Can indeed all mathematically defined unitary operations be achieved (or approximated) as the real evolution processes? A certain monotony of this representation can awake certain doubts. The simplest non-trivial quantum manifold is just a sphere ('qubit'). A pair or n similar qubits is obtained by the tensor product of two of or more spheres. The situation resembles the Ptolemean astronomy in which the perfect form was a circle and the general trajectories were obtained by multiplying circles (thus obtaining the epicycles). If not the astronomic observations which could show the planet trajectories with minimal perturbations from the distant bodies, the Keplerian astronomy would be probably discovered with considerable delay.

Our report is organized as follows. Sec. II presents the 'trajectory doctrine' which permits to deduce the quantum evolution in the quadratic Hamiltonians from the classical motion trajectories. In Sec. III and Sec. IV we present the analogues of optical operations for massive particles. Sec. V describes the squeezing effects in the traditional Paul's traps. Sec. VI discusses the squeezing caused by sharp steps of oscillator potentials. Secs. VII-IX report an elementary case of Toeplitz algebra which permits to design their exact, 'soft' equivalents and Sec. X analyses their possible limitations. Finally, Sec. XI reports some open fundamental challenges.

II The classical-quantum duality

The phenomenon of *squeezing* might seem lateral in quantum mechanics. Yet, it is an essential part of physical experiments, leading to the symplectic transformations of the canonical variables q, p . The simplest cases arise already for non-relativistic time dependent, quadratic Hamiltonians in 2D with variable elastic forces:

$$H(\tau) = \frac{p^2}{2} + \beta(\tau) \frac{q^2}{2} \quad (1)$$

where q, p are the dimensionless canonical position and momentum with $m = \hbar = 1$, $[q, p] = i$ and τ is a dimensionless time coordinate. The question then arises whether the evolution generated by the Hamiltonian (1) can at some moment produce a unitary operator U transforming $q \rightarrow \lambda q, p \rightarrow \frac{1}{\lambda} p$ with $0 \neq \lambda \in \mathbb{R}$, i.e. squeezing q and expanding p or *vice versa*? A more general question is, whether for any pair of observables $a = a^\dagger$ and $b = b^\dagger$, commuting to a number $[a, b] = i\alpha (\alpha \in \mathbb{R})$ an evolution operator can transform $a \rightarrow \lambda a, b \rightarrow \frac{1}{\lambda} b$, i.e. expanding a at the cost of b or inversely?

For nonsingular, bounded $\beta(\tau)$, the evolution equations generated by (1) in both classical and quantum theories lead to exactly the same linear transformations of the canonical variables, represented by

the same family of 2×2 symplectic evolution matrices $u(\tau, \tau_0)$:

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = u(\tau, \tau_0) \begin{pmatrix} q(\tau_0) \\ p(\tau_0) \end{pmatrix}; \quad u(\tau_0, \tau_0) = 1, \quad (2)$$

given by the matrix equations

$$\frac{d}{d\tau} u(\tau, \tau_0) = \Lambda(\tau) u(\tau, \tau_0); \quad \Lambda(\tau) = \begin{pmatrix} 0 & 1 \\ -\beta(\tau) & 0 \end{pmatrix}. \quad (3)$$

The reciprocity between the classical and quantum theories does not end up here. It turns out that, in absence of spin, each unitary evolution operator $U(\tau, \tau_0)$ in $L^2(\mathbb{R})$ generated by the time dependent, quadratic Hamiltonian (??) is determined, up to a phase factor, by the classical motion trajectories. Indeed, it is enough to notice that if two unitary operators U_1 and U_2 produce the same transformation of the canonical variables *i.e.* $U_1^\dagger q U_1 = U_2^\dagger q U_2$ and $U_1^\dagger p U_1 = U_2^\dagger p U_2$, then $U_1 U_2^\dagger$ commutes with both q and p . Hence, it commutes also with any function of q and p . Since in $L^2(\mathbb{R})$ the functions of q and p generate an irreducible algebra, then $U_1 U_2^\dagger$ must be a c-number and since it is unitary, it can be only a phase factor, $U_1 U_2^\dagger = e^{i\varphi} \Rightarrow U_1 = e^{i\varphi} U_2$ where $\varphi \in \mathbb{R}$ [12, 13, 14, 15].

Any two unitary operators which differ only by a c-number phase, even if acting differently on the *state vectors*, generate the same transformation of *quantum states*, so we shall call them *equivalent*, $U_1 \equiv U_2$. It follows immediately that the trajectories of the classical motion problem with the quadratic $H(\tau)$ determine completely the evolution of quantum pure or mixed states $\rho = \rho^\dagger \geq 0$, $\text{Tr } \rho = 1$ and, modulo equivalence, the entire unitary history $U(\tau)$. So, the quantum evolution is defined exactly by the classical motion trajectories. This concerns not only the centers of the packets, but also all higher statistical moments, reducing the problem of evaluating the quantum evolution to the simpler task of integrating the classical motions and using the results to determine the quantum uncertainties.

III Elementary models

Some traditional models illustrate the above "trajectory doctrine". Two of them seem of special interest.

(1) The evolution of charged particles in the hyperbolically shaped ion traps [16]. The Paul's potentials $\Phi(\mathbf{x}, t)$ in the trap interior generated by the voltage $\Phi(t)$ on the surfaces are either $= \frac{e\Phi(t)}{r_0^2} \left(\frac{x^2}{2} + \frac{y^2}{2} - z^2 \right)$ or $\Phi = \frac{e\Phi(t)}{r_0^2} \left(\frac{x^2}{2} - \frac{y^2}{2} \right)$. The problem then splits into the partial Hamiltonians of the type: $H(t) = \frac{p^2}{2} + \frac{e\Phi(t)}{r_0^2} \frac{q^2}{2}$, where q, p represent just one of independent pairs of canonical observables. Now, by introducing the new dimensionless time variable $\tau = \frac{t}{T}$, where T stands for an arbitrarily chosen *time scale*, each 1D Hamiltonian is reduced to a simplified form:

$$\tilde{H}(\tau) = H(t)T = \frac{\tilde{p}^2}{2} + \beta(\tau) \frac{\tilde{q}^2}{2}, \quad \beta(\tau) = \frac{e\Phi(t)T^2}{r_0^2 m}, \quad (4)$$

where $\beta(\tau)$ is dimensionless and the new canonical variables $\tilde{q} = \sqrt{\frac{m}{T}} q$ and $\tilde{p} = \sqrt{\frac{T}{m}} p$ are then expressed in the same units (square roots of the action), leading to the dimensionless evolution matrices $u(\tau, \tau_0)$ identical for the classical and quantum dynamics. So, *without even knowing* about the existence of quantum mechanics, the dimensionless quantities can be now constructed:

$$q_d = \frac{\tilde{q}}{\sqrt{\hbar}} = q \sqrt{\frac{m}{\hbar T}}, \quad p_d = \frac{\tilde{p}}{\sqrt{\hbar}} = p \sqrt{\frac{T}{\hbar m}} \quad (5)$$

and $H_d(t) = \frac{H(t)T}{\hbar}$, where \hbar is an arbitrarily chosen action unit (*cf.* [18]). By knowing already about the quantum background of the theory, an obvious (though not obligatory) option is to choose \hbar as the Planck constant (though the other constants proportional to \hbar of Planck are neither excluded¹). By dropping the unnecessary indexes, one ends up with the evolution problem (??), with an arbitrary time dependent $\beta(\tau)$ (not necessarily coinciding with that of Paul²).

(2) The similar dynamical law applies to charged particles moving in a time dependent magnetic field, given (in the first step of Einstein-Infeld-Hoffmann approximation [17]) by $\mathbf{B}(t) = \mathbf{n}B(t)$, where \mathbf{n} is a constant unit vector (defining the central z -axis of the cylindrical solenoid). Since $\mathbf{B}(t)$ has the vector potential $\mathbf{A}(x, t) = \frac{1}{2}\mathbf{B}(t) \times \mathbf{x}$, the Hamiltonian describing the non-relativistic motion of the charge e in the field $\mathbf{B}(t)$ is expressed (in Gaussian units) by

$$H(t) = \frac{1}{2m} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 = \frac{1}{2m} \left[\mathbf{p}^2 + \left(\frac{eB(t)}{2c} \right)^2 (x^2 + y^2) \right] - \frac{eB(t)}{2mc} M_z. \quad (6)$$

After separating the free motion along the z -axis and the easily integrable rotations caused by $M_z = xp_y - yp_x$ (both commuting with $H(t)$), the motion of the non-relativistic charged, spinless particle on $2D$ plane \mathcal{P}_\perp perpendicular to \mathbf{n} , obeys the simplified Hamiltonian

$$H(t) = \frac{1}{2m} \left[\mathbf{p}^2 + \left(\frac{eB(t)}{2c} \right)^2 \mathbf{x}^2 \right], \quad (7)$$

where \mathbf{p} and \mathbf{x} are the pairs of canonical momenta and positions on \mathcal{P}_\perp . This, after using the dimensionless variables $\tau = \frac{t}{T}$, with x, p_x and y, p_y replacing q, p in (??), leads again to a pair of motions of type (??), with the dimensionless τ, q, p and

$$\beta(\tau) = \kappa^2(\tau) = \left(\frac{eTB(T\tau)}{2mc} \right)^2. \quad (8)$$

In case, if $B(t)$ oscillates periodically with frequency ω the natural dimensionless time $\tau = \omega t$ leads again to a dimensionless $H_d = \frac{H(t)}{\hbar\omega}$, although the stability thresholds no longer obey the Strutt diagram (see [19, 20]).

IV The ‘optics’ of massive particles: classification

In quantum optics of *coherent photon states*, an important role belongs to the *parametric amplification* of Mollow and Glauber [21] (see *e.g.* the encyclopedic work of Dodonov [22]). Yet, in the description of massive particles the Heisenberg’s evolution of the canonical observables (*i.e.*, the “trajectory doctrine”) receives less attention, even though it allows to extend the optical concepts [24, 25]. This can be of special interest for charged particles in the ion traps, driven by the time dependent fields, coinciding or not with the formula of Paul [16]. What most interesting here is the case of quite arbitrary periodic potentials.

¹Indeed, for the quadratic Hamiltonians all results of “Mr. Tompkins in wonderland” by George Gamov, can be deduced just by rescaling time, canonical variables and electric fields.

²Note that the period of the external oscillating field may, but needs not to be used as the ‘reference time’ T to define the dimensionless variable $\tau = \frac{t}{T}$. In his known paper [16] W. Paul assumes the voltage on the trap walls $\Phi(t) = \Phi_0 + \Phi_1 \cos \omega t$, choosing $\tau = \frac{\omega t}{2}$. Here, we assume simply $\tau = \omega t$ and use the Mathieu equation together with the stability diagram of Strut in Bender-Orszag form [19] to compare easily with [20]. More general τ -dependencies will be also considered.

For all periodic fields, $\beta(\tau + T) = \beta(\tau)$, the most important matrices (??) are $u(T + \tau_0, \tau_0)$ describing the repeated evolution incidents (we permitted ourselves to fix the initial point τ_0). Since they are symplectic, their algebraic structure is fully defined just by one number $\Gamma = \text{Tr } u(T + \tau_0, \tau_0)$, (without referring to the Ermakov-Milne invariants [36, 37, 23]). Though the matrices $u(T + \tau_0, \tau_0)$ depend on τ_0 , Γ does not, permitting to classify the evolution processes generated by β in any periodicity interval. The distinction between the three types of behavior is quite elementary:

- I** If $|\Gamma| < 2$ the repeated β -periods, no matter the β -details, produce a stable evolution process. It allows the construction of the global "creation" and "annihilation" operators a^+, a^- defined by the row eigenvectors of $u(T + \tau_0, \tau_0)$, but characterizing the evolution in the whole periodicity interval (compare [14, 20]).
- II** If $|\Gamma| = 2$ the process generated by β belongs to the stability threshold permitting to approximate a family of important dynamical operations including the free evolution acceleration, slowing, or inversion (*cf.* the discussions in [14, 15, 20]).
- III** If $|\Gamma| > 2$ then each single-period matrix $u(T + \tau_0, \tau_0)$ produces the squeezing of a distinguished pair of canonical observables a^\pm defined again by the eigenvectors of $u(T + \tau_0, \tau_0)$, though now associated with a pair of real eigenvalues, $\lambda^\pm = \frac{1}{\lambda^\mp}$; that is, a^+ expands at the cost of contracting a^- or vice versa.

The above global data seem more relevant than the description in terms of the "instantaneous" creation and annihilation operators which do not make obvious the stability/squeezing thresholds. To illustrate all this, it is interesting to integrate (??) for particular case of $\beta(\tau) = \beta_0 + 2\beta_1 \cos \tau$ representing the Paul's potential for (β_0, β_1) in the squeezing areas.

V The 'Mathieu squeezing'

Since the squeezing cannot occur if $\beta(\tau)$ is symmetric in the operation interval [15, 24, 25] we chose to integrate numerically (??) for Paul's $\beta = \beta_0 + 2\beta_1 \cos \tau$ in $[\frac{\pi}{2}, \frac{5\pi}{2}]$ and β_0, β_1 varying in the second squeezing area of the Strutt diagram (compare [19, 20]). We then performed the scanning, to localize the evolution matrices u yielding the position squeezing. The results shown on Fig. 1 generalize the numerical data of Ramirez [20].

The continuous (red) line on Fig. 1 represents the (β_0, β_1) values for which the evolution matrix $u = u(\frac{5\pi}{2}, \frac{\pi}{2})$ has $u_{1,2} = 0$, while the interrupted (blue) line contains the (β_0, β_1) where $u_{2,1} = 0$. The intersection of both yields (β_0, β_1) granting the genuine position-momentum squeezing $q \rightarrow \lambda q$ and $p \rightarrow \frac{1}{\lambda} p$, compare [20], while the additional numbers above the red line show more general effects when q is squeezed (or amplified) at the cost of distinct canonical variables $a^- = u_{2,1} q + \frac{1}{\lambda} p$. Our computer program picked up the four evolution matrices representing various cases of squeezing granted by four pairs (β_0, β_1) on Fig. 1; among them, the u_s approximates the traditional coordinate squeezing at the cost of the corresponding momentum.

$$\begin{aligned} u_1 &= \begin{pmatrix} 0.3625 & 0.0023 \\ -1.1147 & 2.7518 \end{pmatrix}, & u_2 &= \begin{pmatrix} 0.1757 & 0.0053 \\ 3.5018 & 5.7980 \end{pmatrix}, \\ u_3 &= \begin{pmatrix} 0.2161 & 0.0082 \\ 5.4446 & 4.8334 \end{pmatrix}, & u_s &= \begin{pmatrix} 0.227570 & 0.007556 \\ 0.000447 & 4.394266 \end{pmatrix}. \end{aligned} \tag{9}$$

The boring problem of physical units brings, however, some interesting data. For the dimensionless $\tau = \omega t$ the parameter T in (??) is the period of the oscillating Paul's voltage on the trap wall $\Phi(t) = \Phi_0 + \Phi_1 \cos \omega t \Rightarrow \beta(\tau) = \beta_0 + 2\beta_1 \cos \tau$. Hence, the same dimensionless matrix u_s of (??) can be generated in $[\frac{\pi}{2}, \frac{5\pi}{2}]$, by all physical parameters such that:

$$\frac{e\Phi_0}{\omega^2 r_0^2 m} = \beta_0, \quad \frac{e\Phi_1}{\omega^2 r_0^2 m} = 2\beta_1. \quad (10)$$

meaning a considerable scaling freedom. In case of particles with fixed mass and charge, what can vary are the potentials Φ and the physical time T of the operations performed in the dimensionless interval $[\frac{\pi}{2}, \frac{5\pi}{2}]$. If repeated, they are represented by $\omega = \frac{2\pi}{T}$. Hence, for any fixed r_0 , the smaller ω (and the longer the operation time T) the smaller voltages Φ_0 and Φ_1 are sufficient to assure the same result (of course, under the condition that too weak fields do not permit the particle to escape or to collide with the trap surfaces).

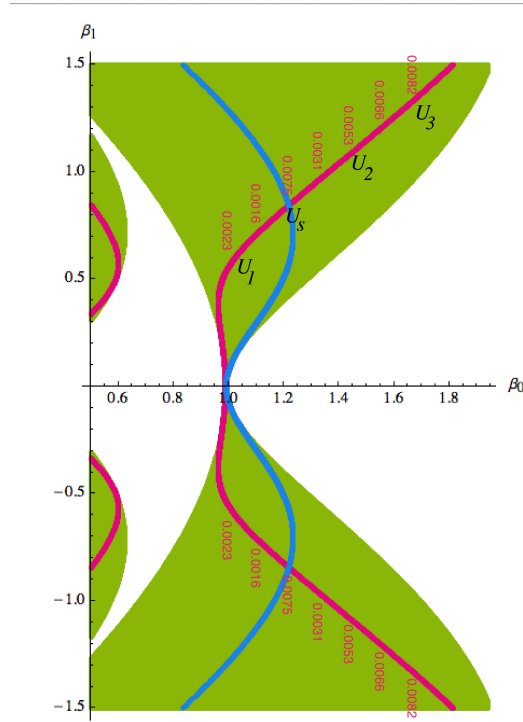


Figure 1: The squeezing in Paul's trap. The red line collects the evolution matrices (??) with $u_{1,2} = 0$ in the second squeezing area on the Strutt map. The numbers $\lambda = u_{1,1}$ above, define the *squeezing* of q completing the data obtained in [20]. The particular matrices u_1, u_2, u_3 and u_s obtained for $(\beta_0, \beta_1) = \{(1.054, 0.646), (1.577, 1.231), (1.774, 1.454), (1.217, 0.844)\}$, respectively, are reported in formula (??).

For a proton ($m = m_p \simeq 1.67 \times 10^{-24} \text{g}$) in an unusually ample ion trap of $r_0 = 10 \text{cm}$ and in a moder-

ately oscillating Paul's field with ω corresponding to a 3km long radio wave, one would have $\omega^2 r_0^2 m_p \simeq 10^{-12} \text{g} \frac{\text{cm}^2}{\text{s}^2} = 1.67 \times 10^{-12} \text{g} \frac{\text{cm}^2}{\text{s}^2} \simeq 1.04233 \text{eV}$, leading to the voltage estimations: $\Phi_0 \simeq 1.0423 \beta_0 V \simeq 1.268 V$ and $\Phi_1 \simeq 2.0846 \beta_1 V \simeq 1.759 V$. In a still wider trap (of an S/F size!) $r_0 = 100 \text{cm}$ or, alternatively, for $r_0 = 10 \text{cm}$ but the frequency 10 times higher, the voltages needed on the walls should be already 100 times higher.

While the existing analytic expressions [26] still did not offer the exact results, the above computer experiment shows that the phenomena of coordinate squeezing indeed can happen in the Paul's traps. Yet, our computer data describe only the hypothetical evolution in extremely 'clean' field oscillations, without any laser cooling, nor any dissipative perturbations. Moreover, the squeezing effects described by matrices (??) are volatile, materializing itself only in sharply defined time moments, which makes difficult the observation of the phenomenon in the oscillating trap fields.

VI The option of squeezed Fourier

Mathematically, the simplest way to construct the quantum operations is to apply sequences of δ -pulses interrupting some continuous evolution process (e.g. the free evolution, the harmonic oscillation, etc. [15, 27, 28, 29, 30]), though they are obviously limited by the difficulty of applying the sharp pulses of external fields. In case of squeezing, a more regular method could be to superpose some evolution incidents which belong to the equilibrium zone (I) but their products do not. One of chances is to use the fragments of time independent oscillator Hamiltonians (??) with the elastic forces $\beta = \kappa^2 = \text{const}$, generating the 'symplectic rotations':

$$u = \begin{pmatrix} \cos \kappa \tau & \frac{\sin \kappa \tau}{\kappa} \\ -\kappa \sin \kappa \tau & \cos \kappa \tau \end{pmatrix}. \quad (11)$$

Their simplest fragments obtained for $\cos \kappa \tau = 0$ are the *squeezed Fourier transformations*

$$u = \begin{pmatrix} 0 & \pm \frac{1}{\kappa} \\ \mp \kappa & 0 \end{pmatrix}. \quad (12)$$

Following the proposal of Fan and Zaidi [31] and Grübl [32] (cf. also [18]) it is enough to apply two such steps with different κ -values to generate the evolution matrix:

$$u_\lambda = \begin{pmatrix} 0 & \pm \frac{1}{\kappa_1} \\ \mp \kappa_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \pm \frac{1}{\kappa_2} \\ \mp \kappa_2 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}; \quad \lambda = -\frac{\kappa_2}{\kappa_1} \quad (13)$$

which produces the squeezing of the canonical pair: $q \rightarrow \lambda q, p \rightarrow \frac{1}{\lambda} p$, with the effective evolution operator: $U_\lambda = \exp[-i\sigma \frac{pq+qp}{2}]$; $\sigma = \ln \lambda$. It requires, though, two different $\kappa_1 \neq \kappa_2$ in two different time intervals divided by a sudden potential jump. (Here, the times τ_1 and τ_2 must fulfill e.g. $\kappa_1 \tau_1 = \kappa_2 \tau_2 = \frac{\pi}{2}$ to assure that both κ_1 and κ_2 grant two distinct squeezed Fourier operations in their time intervals.) If one wants to apply two potential steps on the null background, it means at least three jumps ($0 \rightarrow \kappa_1 \rightarrow \kappa_2 \rightarrow 0$). How exactly can one approximate a jump of the elastic potential? Moreover, each κ -jump implies an energy transfer to the microparticle (cf. Grübl [32]). Yet, could the pair of generalized Fourier operations in (??) be superposed in a soft way with an identical end result? In fact, the recent progress in the inverse evolution problem shows the possibility of such exact effects.

VII The soft operations

The method employs the simplest *algebra of Toeplitz* of 2×2 *equidiagonal, symplectic matrices* u with $u_{11} = u_{22} = \frac{1}{2}\text{Tr } u$. It turns out that if two matrices u, v are equidiagonal, then their anticommutator $uv + vu$ and the symmetric products uvu and vuv have the same property. The *Toeplitz matrices* were studied by mathematicians [33, 34], though apparently, without noticing their full quantum control sense. In our case, they do not necessarily eliminate jumps in (??) but they give more flexibility in constructing the squeezed Fourier operations which now can be achieved as the symmetric products of many little symplectic contributions (??) with different β 's acting in different time intervals. Thus, *e.g.*, by using the fragments of the 'symplectic rotations' v_k caused by the Hamiltonians (??) with some constant $\beta = \beta_k$ in time intervals $\Delta\tau_k$ ($k = 0, 1, 2, \dots$), one can define the symmetric product:

$$u = v_n \dots v_1 v_0 v_1 \dots v_n \quad (14)$$

which is again symplectic and *equidiagonal* (*i.e.* of the simplest Toeplitz class), with $u_{11} = u_{22} = \frac{1}{2}\text{Tr } u$. Whenever (??) achieves $\text{Tr } u = 0$, the matrix u becomes *squeezed Fourier* too. The continuous equivalents can be readily obtained. Indeed, it is enough to assume that the amplitude $\beta(\tau)$ is symmetric around a certain point $\tau = 0$, *i.e.*, $\beta(\tau) = \beta(-\tau)$. By considering then the limits of little jumps du caused by applying the contributions $dv = \Lambda(\tau)d\tau$ from the left and right sides, one arrives at the differential equation for $u = u(\tau, -\tau)$ in the expanding interval $[-\tau, \tau]$:

$$\frac{du}{d\tau} = \Lambda(\tau)u + u\Lambda(\tau). \quad (15)$$

Its anticommutator form leads easily to an exact solution. Since $\Lambda(\tau)$ is given by (??), the equation (??) becomes

$$\begin{aligned} \frac{du}{d\tau} &= \begin{pmatrix} u_{21} - \beta u_{12} & \text{Tr } u \\ -\beta \text{Tr } u & u_{21} - \beta u_{12} \end{pmatrix} \\ &= (u_{21} - \beta u_{12})\mathbb{1} + \text{Tr } u \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix} \end{aligned} \quad (16)$$

(*cf.* [15, 20]). For the symmetric $\beta(\tau)$, this determines explicitly the matrices $u = u(\tau, -\tau)$ for the expanding $[-\tau, \tau]$ in terms of just one function $\theta(\tau) = u_{12}(\tau, -\tau)$. In fact, since (??) implies the same differential equation for u_{11} and u_{22} , *i.e.* $\frac{du_{11}}{d\tau} = \frac{du_{22}}{d\tau} = u_{21} - \beta u_{12}$, and $u_{11} = u_{22} = 1$ at $\tau = 0$, then $u_{11} = u_{22} = \frac{1}{2}\text{Tr } u = \frac{1}{2}\theta'(\tau)$ for u of any symmetric $[-\tau, \tau]$. Moreover, since $u = u(\tau, -\tau)$ are symplectic, *i.e.* $\text{Det } u = \left[\frac{1}{2}\theta'(\tau)\right]^2 - \theta u_{21} = 1$, one obtains

$$u_{21} = \frac{\left[\frac{1}{2}\theta'(\tau)\right]^2 - 1}{\theta} \quad (17)$$

Hence, (??) defines the amplitude $\beta(\tau)$ which had to be applied to create the matrices $u = u(\tau, -\tau)$. Indeed:

$$\beta u_{12} = u_{21} - \frac{du_{11}}{d\tau} \quad (18)$$

and since $u_{12} = \theta$, the $\frac{du_{11}}{d\tau} = \frac{\theta''}{2}$ and u_{21} is given by (??), then:

$$\beta = -\frac{\theta''}{2\theta} + \frac{\left[\frac{1}{2}\theta'(\tau)\right]^2 - 1}{\theta^2} \quad (19)$$

This solves the "symmetric evolution problem" for u and β in any interval $[-\tau, \tau]$ in terms of one function $\theta(\tau)$, obeying some non-trivial conditions in single points only.

Lemma. Suppose $\beta(\tau)$ is continuous and symmetric around $\tau = 0$, in a certain interval $[-T, T]$, given by (??), where $\theta(\tau)$ is continuous and three times differentiable. The conditions which assure the continuity and differentiability of β are then:

- i At any point τ where $\theta(\tau) = 0$, there must be $\theta'(\tau) = \pm 2$.
- ii At any point τ where $\theta(\tau) \neq 0$ but $\theta'(\tau) = 0$, the matrix (??) of $[-\tau, \tau]$ represents the squeezed Fourier transformation with the amplitude $\beta(\tau)$ at the end points given by $\beta(\tau)\theta^2 = -\frac{1}{2}\theta''\theta - 1$. If, moreover, $\theta'''(\tau) = 0$ then also $\beta'(\tau) = 0$.

Proof follows straightforwardly by applying (??). In particular, since (??) and the initial condition grants $u_{11} = u_{22} = \frac{1}{2}\theta'(\tau)$ then, whenever $\theta' = 0$, both $u_{11} = u_{22} = 0$ implying $u_{12} = b \neq 0, u_{21} = -\frac{1}{b}$; which is the general form of the *squeezed Fourier* transformation. Simultaneously, (??) simplifies and the value of $\beta(\tau)$ fulfills $\beta(\tau)\theta^2 = -\frac{1}{2}\theta''\theta - 1 \Rightarrow \beta(\tau)b + \frac{\theta''}{2} + \frac{1}{b} = 0$. In particular, if $\theta''(\tau)b = -2$, then $\beta(\tau) = 0$.

Certain curious *quid pro quo* should be registered. By ignoring the adiabatic techniques [40, 41, 42] we used here (and in [14, 15, 20]) the simplest case of Toeplitz algebra (see [33, 34]) which solves the inverse evolution problem for $\beta(\tau) = \kappa^2(\tau)$ in terms of $\theta(\tau)$ without any auxiliary invariants. However, its purely comparative sense should be stressed. Fixed a pair of canonical variables q, p it does not give the causally progressing steps of the evolution, but rather compares the 'evolution incidents' in a family of expanding intervals $[-\tau, \tau]$. Should we follow the causal development of our classical/quantum systems, the Ermakov-Milne equation [36, 37] might be useful. An interrelation between both methods waits still for an exact decoding. It is not excluded that the anticommutator algebras can also help to solve some higher dimensional canonical evolution problems.

VIII The simplest cases

As already checked, there exist polynomial models of $\theta(\tau)$ [15] making possible the soft generation of the squeezed Fourier transformations (no sudden jumps!). The polynomials, however, are just a formal exercise. In fact, the experimental techniques of producing an arbitrarily time dependent fields in Paul's traps still wait to be elaborated. As it seems, empirically more natural would be to apply the harmonically oscillating θ functions. As the most elementary case, let us consider the evolution guided by θ' as with only three frequencies. In dimensionless variables:

$$\theta(\tau) = a_1 \sin \tau + a_3 \sin 3\tau + a_5 \sin 5\tau. \quad (20)$$

Note that for $\theta(\tau)$ antisymmetric, the corresponding $\beta(\tau)$ defined by (??) is symmetric around $\tau = 0$. The conditions of our lemma for the θ -function given by (??) to generate softly "the squeezed Fourier" with $u_{12} = \pm \kappa = b$ at the ends of the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ reduce then to:

$$\begin{aligned} \theta'(0) &= a_1 + 3a_3 + 5a_5 = 2, \\ \theta\left(\frac{\pi}{2}\right) &= a_1 - a_3 + a_5 = b, \\ \theta''\left(\frac{\pi}{2}\right) &= -a_1 + 9a_3 - 25a_5 = -\frac{2}{b} - 2b\beta_0, \end{aligned} \quad (21)$$

where the first identity grants the nonsingularity of β in 0, the second one defines the magnitude b of the *Fourier squeezing*, and β_0 defines the symmetric values of the amplitude $\beta(\tau)$ at $\pm \frac{\pi}{2}$. The equations (??) are then fulfilled by:

$$a_1 = \frac{1}{16} \left(4 - \frac{2}{b} + b(15 - 2\beta_0) \right), \quad a_3 = \frac{1}{32} \left(12 - \frac{2}{b} - b(5 + 2\beta_0) \right), \quad a_5 = \frac{1}{32} \left(\frac{2}{b} + 4 - b(3 - 2\beta_0) \right), \quad (22)$$

with $b \neq 0$ an arbitrary real constant. Note that our assumed (antisymmetric) $\theta(\tau)$ has the physical sense of $u_{12}(\tau, -\tau)$ for $\tau > 0$, while the obtained (symmetric) $\beta(\tau)$ represents the field amplitude in a certain symmetric τ -interval. The three symmetric amplitudes $\beta(\tau)$ with $b = 2$, $b = \frac{5}{3}$ and $b = \frac{184}{95}$ in the operation interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ are represented on Fig. 2 below.

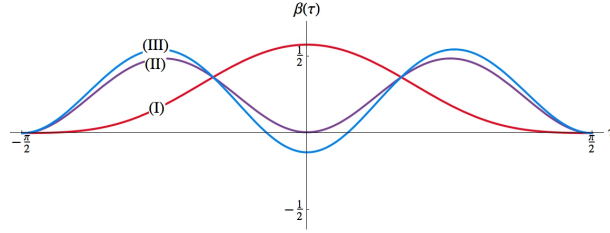


Figure 2: The form of three symmetric β -amplitudes vanishing softly on the borders of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with (I) $b = \frac{5}{3}$, (II) $b = \frac{184}{95}$, (III) $b = 2$; all of them assuring the generation of the squeezed Fourier by the Hamiltonian (??). Both (I) and (II), which do not cross to the negative values, are adequate to achieve the "squeezed Fourier" operations by time dependent magnetic fields.

Of course, the θ and β -pulses determined by (??) and (??), shown on Fig. 2 are a kind of 'extra information' which tells only what operations the time dependent β -functions must generate at the ends of both symmetric operation intervals (in this case the pair of 'squeezed Fourier' in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{2}]$). This does not yet define the actual trajectory inside both intervals (*i.e.* for $\tau \neq -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$), which must be determined by a separate computer simulation. With this aim, we integrated the matrix equation (??) (not (??)!) for $u(\tau, \tau_0)$ with the initial condition $u(\tau_0, \tau_0) = \mathbb{1}$, where $\tau_0 = -\frac{\pi}{2}$ starts the first evolution interval, then we repeated the same with the next β in the next interval, obtaining a family of 2×2 evolution matrices which draw a congruence of trajectories departing from the beginning of the first and ending up at the end of the second evolution interval (c.f 3 (a)). As one can see, they indeed paint an image of the 'squeezed Fourier' at the operation center (in $\tau = \frac{\pi}{2}$) and the coordinate squeezing at the end.

The above time dependent family $u(\tau, \tau_0)$ in the sum of both intervals $[-\frac{\pi}{2}, \frac{3\pi}{2}]$, permits also to determine the progress of the position and momentum uncertainties on the trajectory. As an example, we took one of the the most elementary Gaussian wave functions in $L^2(\mathbb{R})$ centered at $x = q_0$ with the initial velocity p_0 :

$$\Psi = A e^{i p_0 (x - q_0)} e^{-\kappa \frac{(x - q_0)^2}{2}}, \quad A = \left(\frac{\kappa}{\pi} \right)^{\frac{1}{4}} \quad (23)$$

For $\kappa = 1 = q_0 = 1$, for varying p_0 the packet center will draw exactly the family of trajectories on Fig. 3 and the simple calculation with the initial uncertainties $(\Delta q)^2 = (\Delta p)^2 = \frac{1}{2}$, leads to:

$$[\Delta q(\tau)]^2 = \frac{1}{2} [u_{11}^2(t) + u_{12}^2(t)], \quad (24)$$

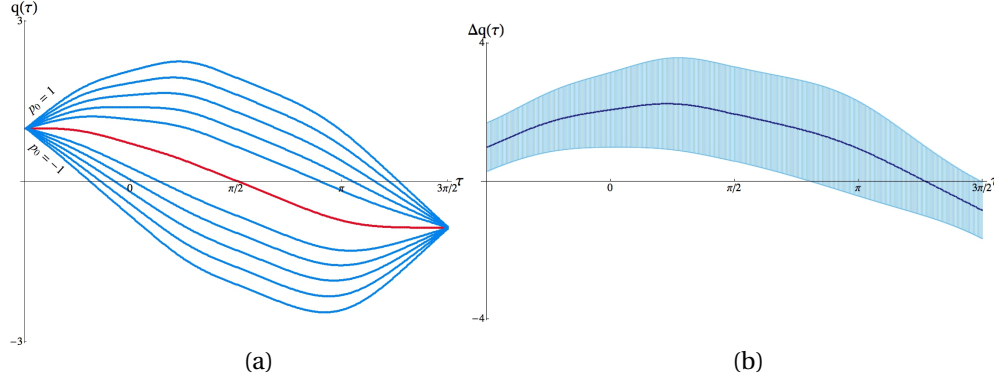


Figure 3: (a) The congruence of the charged trajectories in dimensionless variables generated by the pair of the squeezed Fourier operations induced by two 'soft' β -amplitudes of Fig. 2 for $b_I = \frac{5}{3}$ and $b_{II} = \frac{184}{95}$ in two subsequent intervals $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{2}]$. The final result is the q -amplification with $\lambda = -1.16211$. If generated by a magnetic field in a cylindrical solenoid, it would mean the λ -expansion of both coordinates $q = x, y$. (b) The position uncertainty Δq (the 'uncertainty shadow') for the upper trajectory on the part (a) was determined for the numerically calculated u_{11} and u_{12} according to (??) with the initial values $\Delta q = \Delta p = \frac{1}{2}$. So, if the operation is performed for an initial Gaussian packet, in a cylindric Paul's trap or solenoid of dimensionless radius large enough (e.g. $r_0 > 10$), it implies a little probability of the particle collision with the trap or solenoid wall.

We then used the square root of (??) to correct the upper trajectory of Fig. 3(a) ($p_0 = 1$) by its *uncertainty shadow* (see Fig. 3(b)). These results suggests that the main part of the evolving packet is contained within a wider dimensionless belt, e.g. $|q(\tau)| < 10$ in the whole evolution interval $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. Characteristically, the uncertainty effects are most visible in the middle of the trajectory, for $\tau = \frac{\pi}{2}$ where two distinct 'squeezed Fourier' meet, but they obey the final 'amplified state' at $\tau = \frac{3\pi}{2}$.

We deliberately choose the case of both β -amplitudes starting and ending up with $\beta(-\frac{\pi}{2}) = \beta(\frac{\pi}{2}) = \beta(\frac{3\pi}{2}) = 0$ to illustrate the flexibility of the method. In fact, we could notice that some programs of 'frictionless driving' of quantum states seems to be limited to the bound state Hamiltonians (like (??)) at the beginning and at the end of the transport operation. Thus, e.g., in an interesting report Xi Chen *et al* [41] the authors present an operation modifying the harmonic oscillator H_0 by adding some perturbation H_1 , which vanishes before and after the operation, it can also appear or disappear suddenly (likewise in [40]). The interruption of an adiabatic process by a new potential which can suddenly "jump to existence" seems, however, a questionable point of the method. Even if the H_0 is an oscillator potential, the difficulty consist in the sudden jump (see the consequences proved by Grübl [32]).

We thus took an occasion to check that the squeezing/amplification of the wave packets can be as easily produced by just modifying slightly the orthodox harmonic potential $\beta(\tau)$ in the middle of the operation interval but conserving the same $\beta = \beta_0 > 0$ at both ends. An example is presented on our Fig. 4 where the three variable amplitudes $\beta(\tau)$, all of them generating the squeezed Fourier operations in $\beta(-\frac{\pi}{2}) = \beta(\frac{\pi}{2})$, are softly unified with the constant β_0 of the time independent harmonic oscillator in

$\left[\frac{\pi}{2}, \frac{5\pi}{2\sqrt{7}}\right]$ (we took care to preserve the continuity of β , β' and β'' at $\frac{\pi}{2}$) The amplification (or squeezing) caused by the red amplitude (I) turns softer than for the amplitudes of Fig. 4, yielding now $\lambda \simeq 1.053$. The uncertainty shadow is again illustrated on part (b).

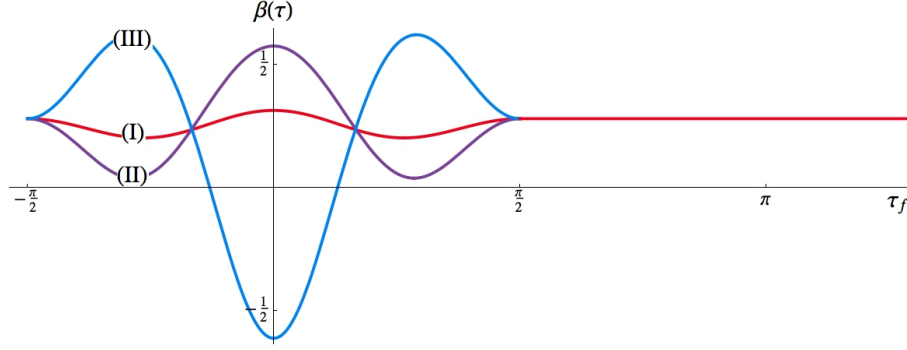


Figure 4: The β -amplitudes satisfying (??) with the initial $\beta_0 = \frac{7}{25}$ and: (I) with $b = \frac{199}{100}$, (II) with $b = \frac{37}{20}$, (III) with $b = \frac{5}{2}$. Analogous to Fig. 3, both (I) and (II) pulses grant the "magnetic squeezed Fourier" in their first action interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In the next τ -interval $[\frac{\pi}{2}, \frac{5\pi}{2\sqrt{7}}]$ they all reduce themselves to the constant β_0 generating the same squeezed Fourier with $b_0 = \frac{5}{\sqrt{7}}$. The fragments of (I) in both intervals produce amplification/squeezing with $\lambda \simeq 1.07...$

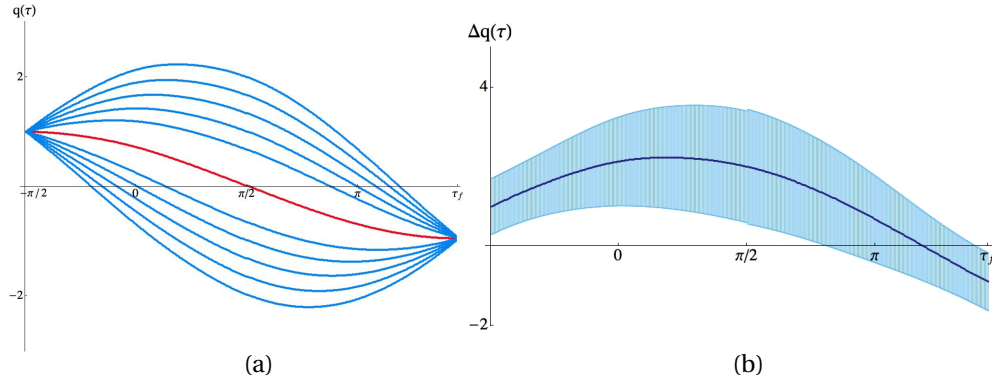


Figure 5: The soft amplification/squeezing with $\lambda \simeq 1.07$ generated by the amplitude (I) of Fig.4 : (a) classical trajectories, (b) the evolving 'shadow of uncertainty', on both parts of the amplitude (I).

All this were just the simplest 1D options of generating the soft squeezing of q and p . In the axially symmetric Paul's traps this would lead to the simultaneous squeeze (expansion) of both Cartesian coordinates on the 2D plane. The data on Figs. 3-5 are applicable also to the magnetic pulses in cylindrical

solenoids where, after the separation of the easily integrable rotations, the motion obeys the two dimensional "magnetic oscillators" on the plane orthogonal to the magnetic fields $\mathbf{B}(t)$. If only the dimensionless amplitude $\beta(\tau)$ of (??) coincides with the positive amplitudes of Fig. 2, then the pair of the squeezed Fourier operations will produce an amplification of $\mathbf{q} = (x, y)$ at the cost of squeezing $\mathbf{p} = (p_x, p_y)$ (or inversely). A certain surprise are the extremely delicate values of the squeezing effects and the corresponding electric and magnetic fields (*cf.* Table 1). Can so weak interactions keep the particle and dictate its unitary transformations? Without entering deeper into the discussion, let us only notice that the extremely weak fields could be of tentative importance in our own existence [51, 52, 53].

IX The orders of magnitude

Since the real operation time τT can be arbitrarily large, and fields arbitrarily weak, so borrowing the terminology from [43, 44], we might call it an *adiabatic squeezing*. It seems to confirm the existence of the squeezing as purely quantum mechanical phenomenon. The fields and times needed in our operations are gathered in Table 1 below.

T [sec.]	1/1000	1	100
q [cm]	0.0008	0.025	0.25
p [g cm sec. ⁻¹]	1.3×10^{-24}	4.2×10^{-26}	4.2×10^{-27}
ν [cm sec. ⁻¹]	0.8	0.025	0.0025
Φ_{\max} [Volt]	98×10^{-3}	98×10^{-9}	9.8×10^{-12}
B_{\max} [G]	160×10^{-3}	160×10^{-6}	1.6×10^{-6}
$F_{\text{rad}}/F_{\text{osc}}$	7.7×10^{-9}	7.7×10^{-12}	7.7×10^{-14}

Table 1: The physical conditions for the amplification/squeezing $\lambda \simeq 1.05$ ($\frac{1}{\lambda} \simeq 0.95$), for a proton in a (hypothetical) axially symmetric Paul's trap with $r_0 = 20\text{cm}$, or in a cylindrical solenoid, generated by soft pulses represented on Fig. 4 and Fig. 5, corresponding to three physical times T . The physical magnitudes of q, p, ν in the upper rows correspond to the dimensionless $q_d = p_d = 1$ in definition (??) for various T . The fields are extremely tiny but their amplitudes grow as the operation time becomes shorter. Orders of magnitude of Φ_{\max} and B_{\max} for the subsequent operation times are quoted in 5th and 6th rows. The last row reports the average ratios of the Abraham-Lorentz radiative force to the time dependent oscillator forces in all operations.

Of course, if both squeezed Fourier operations, given by two different matrices u and v , are many times repeated forming a sequence $uvuvuv\dots$ then they will cause a sequence of the state amplifications growing as $\lambda, \lambda^2, \lambda^3, \dots$ interpolated by increasing sequence of the inverse (squeezing) effects

$\lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \dots$ So, even if the initial effects were very weak ($|\lambda| \simeq 1$), and if the empirical conditions permit, they could be step by step amplified.

Of course, all this are merely the preliminary evaluations valid only for a restricted class of the dimensionless variables (and for the size of traps or solenoids sufficient in physical units). If we are interested in so weak fields, it is not just hoping to develop a new branch of quantum technology, but rather to check the reality of QM states and operators on the most basic level, of delicate, semi classical and quasi-static operations which could verify *e.g.* the quantum mechanical state structure together with its linearity unpolluted by radiative phenomena. To observe this, it is of course insufficient to limit attention to the slowly evolving but strong external fields, as the observation of a microparticle evolution in a strong but static oscillator potential will not bring information about the continuous evolution law of the quantum state in the external forces, but will reveal rather the quantum jumps and instabilities. Hence, our present subject are the external fields which are adiabatically changing but simultaneously weak enough, to keep the quantum system for a long time on purely quantum mechanical level.

The question which still remains of possible importance even for weak adiabatic fields is the radiative pollution caused by the driven object itself. Even if the potential pulses are periodic, the formalism of Floquet Hamiltonians do not seem adequate to estimate the effects. In spite of all doubts (the run-away solutions etc.), we decided to check the possible magnitude of the Abraham-Lorentz radiative force (which seems quite natural in the 'trajectory doctrine'). While the ordinary force of the variable oscillator trajectory is simply $\mathbf{F} = m\ddot{\mathbf{x}}$, the hypothetical radiative force is expressed as $\mathbf{F}_{\text{rad}} = m\sigma\ddot{\mathbf{x}}$, where σ is the particle dependent 'characteristic time' [35]. Using now the definitions of dimensional quantities (??) we can compare the magnitudes of the conventional and radiative forces for the squeezing operations in our table, finding the radiative ones extremely small but slowly increasing for shrinking physical time T (*i.e.* higher frequencies) on our scale (*cf.* the lowest line in Table 1). Further problems in the 'trajectory approach' are still open.

We note that in works using the Ermakov-Milne invariants the operations are supposed to be faster and still 'frictionless' [40, 41, 42], though the authors use the 'invariants' not attending explicitly the radiative friction. Of course, the intuition of the transitionless driving which transports the states without changing the eigenvalues (of relevant invariants) is quite suggestive; maybe, it could still reduce the radiative pollution? Perhaps, our exact solutions, obtained in terms of Toeplitz parameters, are hardly a beginning? Could the more complex variational methods, following *e.g.* [38, 39, 41] be applied at the level of the driving θ -function? The question whether Ermakov can help Toeplitz or *vice versa* is still open³. Some other empirical questions can neither be dismissed.

X Imperfections and open problems

The troubles with geometry. In many labs the techniques used to keep and cool the ions are adequate to study the atomic structures but insufficient to our purposes, since the time dependent oscillator potential is created only in a strictly local scale (*e.g.* in an immediate vicinity of the central axis of a 'quadrupole trap', formed by four metal bars [54]), or in the recently described 'charged resonator' [55], where it is well approximated only in a vicinity of a single point. The technical chance to approach the

³The question is not indeed limited to microobjects. A macroscopic analogue would be a skillful waiter running with a plate full of liquid without spilling a drop.

oscillator fields in a wider space to control the unitary evolution described in the Sec. II occurs either in the conventional or in cylindrical Paul's traps with perfectly hyperbolic surfaces or else, in the interiors of the cylindrical solenoids, in both, if the operation area is wide enough. If so, however, then the field propagation in the trap interior becomes essential.

The relativistic corrections. The finite propagation of the electromagnetic signals in Paul's traps, affecting the scalar potentials on the trap surfaces, in general, are known only with accuracy up to $\frac{1}{c}$ (*post Newtonian*) terms in the Einstein-Infeld-Hoffman (EIH) slow motion approximation [17]). The chances to use the quasi static $\simeq \frac{1}{c^2}$ (*post-post Newtonian*) conditions were considered in [15]. Below, we shall consider the similar problem for softly changing magnetic fields. Of course, the time dependent, homogeneous magnetic field $B(t)$ in the cylindric solenoid does not fulfill the Maxwell equations. Yet, it coincides with the first step of the EIH approximation. To evaluate the errors, let us look for the exact time dependent vector potentials of the solenoid in the form:

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{2}B(r, t)\mathbf{n} \times \mathbf{x} = \frac{1}{2}B(r, t) \begin{pmatrix} -y \\ x \end{pmatrix}, \quad (25)$$

where the magnetic field B instead of depending only on t , could also depend on radius r on the perpendicular solenoid section. To assure the relativistic sense of (??) we must assume $\square \mathbf{A} = \frac{4\pi}{c} \mathbf{j} = 0$ where $\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$. As easily seen, the application of the Laplacian to the right hand side of (??) reduces to apply the operator $D \equiv \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r}$ to $B(r, t)$ alone. Hence, the vanishing of the d'Alambertian $\square \mathbf{A}$ means:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - D \right] B(r, t) = 0. \quad (26)$$

To assure the analytic shape of $B(r, t)$ around the z -axis we look for the solution in the form:

$$B(r, t) = B_0(t) + B_2(t)r^2 + B_4(t)r^4 + \dots \quad (27)$$

where $B_0(t) = B(t)$ is the homogeneous quasistatic approximation. Since $Dr^{2n} = 4n(n+1)r^{2(n-1)}$, (??) and (??) after short calculation yield:

$$B_{2n}(t) = \frac{1}{4^n n!(n+1)!} \frac{1}{c^{2n}} \frac{\partial^{2n}}{\partial t^{2n}} B(t). \quad (28)$$

By introducing now a dimensionless time $\tau = \frac{t}{T}$, where T is some conventional time unit corresponding to the laboratory observation, and by writing (??) in terms of the time derivatives $\frac{\partial}{\partial \tau}$ one can reduce it to:

$$B(r, t) = B(t) + \frac{1}{8} \left(\frac{r}{cT} \right)^2 \frac{\partial^2 B(t)}{\partial t^2} + \frac{1}{24} \left(\frac{r}{cT} \right)^4 \frac{\partial^4 B(t)}{\partial t^4} + \dots \quad (29)$$

We kept here the $B(t)$ depending on the real time t , in order to assure that all derivatives $\frac{\partial^n}{\partial \tau^n} B(t)$ will be expressed in magnetic field units. The curious property of this formula is the absence of terms $\sim \frac{1}{c}$; the field propagation law (??) is solved exclusively in terms of extremely small contributions $\sim \frac{1}{c^2}$. It means that the superpositions of delicate wave fronts running towards the solenoid center create a good approximation of the quasistatic theory.

The control of the currents. The unsolved problem is also how to create (or at least approximate) the homogeneous magnetic field $B(t)$ which does not depend on z , but depends on time in any desired way.

In the static case, the magnetic field B in the solenoid can be achieved with a good accuracy by the stationary current \mathbf{j} circulating around the solenoid surface. The well known result obtained by integrating the magnetic field along the closed contour surrounding the solenoid wires tells that \mathbf{B} is defined by total current circulating per unit of the z -axis, *i.e.*, $B = \frac{4\pi}{c} \frac{\Delta I}{\Delta z}$. These fields are not strong but, tentatively, of interest for the control problems. However, how to produce the circulating currents depending on time, but homogeneous on all surface sections (*i.e.* independent of z)? To create such quasistatic environment, it is still an unsolved laboratory challenge. If the solenoid was constructed as a single spiral wire around the cylindrical surface, connected at the both extremes to the potential difference $\Phi(t)$, then even soft changes of Φ will propagate along the solenoid as a pulse of the current, creating inside the soft but z -dependent fields, instead of the quasistatic $B(t)$. Perhaps, a good approximation could be – instead of using a single wire – to cover the cylindric surface by some number of shorter wires connected with a common source of variable voltage? Could this help? Yet, this seems not the only technical solution.

The model of rotating cylinder. The idea is inspired by a slightly peculiar example described by Griffiths [45]. The cylindrical surface of non-conducting material (*e.g.* glass), and radius R is charged uniformly by surface density σ , so that each circular belt of 1cm height contains the charge of $R\sigma$. The experimental challenge is not extraordinary. If the cylinder has the radius $R = 20\text{cm}$ and rotates with frequency $\omega = 1\text{s}^{-1}$ around its axis z and has the charge 1C on each 1cm horizontal belt, then it will produce inside the homogeneous magnetic field \mathbf{nB} of intensity:

$$B = \frac{4\pi}{c} \omega R \sigma \simeq 1.25\text{G}. \quad (30)$$

with accuracy up to the post-post-Newtonian corrections. Hence, by employing the softly changing angular velocity $\omega = \omega(t)$ one can generate the practically homogeneous magnetic field $\mathbf{nB}(t)$ of the quasistatic environment described by (??). Will this approximation work?

The time control. In order to perform an operation induced by variable fields on the state of a microobject, the microobject must be submitted to the action of these fields in the exact time interval between the operation beginning and the operation end. In case of an operations induced by the time dependent magnetic fields, it means that the charged particle must be injected in the known initial state to the solenoid at the exact beginning of the "squeezed Fourier" operations represented on Fig. 2 and the result of the operation must be checked again in the well defined moment, after several applications of the "soft" pattern. (The need of this double synchronization is almost never considered in papers on quantum control).

What one can imagine is a long but finite solenoid; then the particle injected at a precisely controlled moment, at a point $z = z_0$ in one solenoid end, with a certain velocity v_z . Now, the particle wave packet propagates, changing its shape on the subsequent perpendicular solenoid sections, until arriving to the other end, during the time corresponding exactly to one or more squeezing operations. Once arriving there, it is received by a measuring device (like a photographic plate) registering its position on the new orthogonal plane. Certain errors of this scheme are inevitable. In fact, if the registering screen has a granular structure, then the final particle position will be known only with accuracy to the distance between the mesoscopic detectors. Moreover, since for a particle injected at a given z_0 (solenoid beginning), the velocity of flight along the z -axis must obey the uncertainty relation $[z, p_z] = i\hbar$ so, the time of flight might fail to reproduce exactly the time of the squeezing operation. An interesting aspect is, however, that if the squeezing between the initial and final particle state on two orthogonal planes consists in coordinate amplification, *i.e.*, $\tilde{x} = \lambda x_0$ and $\tilde{y} = \lambda y_0$ and the final momenta shrinking, $\tilde{p}_x = \frac{1}{\lambda} p_x$ and

$\tilde{p}_y = \frac{1}{\lambda} p_y$ (with $|\lambda| > 1$), then the situation is almost equivalent to the non-demolishing measurement [46, 47], but with an important numerical difference, that even from an imprecise measurement of the final coordinates \tilde{x}, \tilde{y} , the data about the initial ones x_0, y_0 will be recovered with much smaller errors $(\Delta x_0, \Delta y_0) = \frac{1}{|\lambda|} (\Delta \tilde{x}, \Delta \tilde{y})$.

The neglected perturbations. In all our calculations we considered the pure particle states, evolving in a slowly changing external fields, without taking corrections for the traces of matter in the ion traps or in the solenoids. So, how perfect must be the trap vacuum, to realize indeed our squeezing operations? Moreover, we have neglected the possible direct packet reflection from or absorption by the walls of the laboratory. So, how large must be the ion trap or the solenoid to make any of these corrections insignificant? We note only that in case of possible particle absorption by a surface, the problem indeed leads to the fundamental question of "time operator" which persists without a truly convincing solution even in case of the flat surfaces. We can hope only that for the traps wide enough, our *soft operations* bring something of interest to the quantum control theory. We also did not consider the variety of unavoidable modifications described by the dissipative mechanisms of Lindblad [48, 49, 50]. Be all this be avoided, some other controversial aspects remain to be discussed.

XI The fundamental aspects 'in little'

What is still missing is a kind of *what if story*. The problem arises, whether the existing difficulties to achieve the squeezing are purely technical or they mean some fundamental barrier? If no barrier exists, and the operations can be indeed performed (or at least approximated), the consequences could be worth attention.

If the squeezing of the wave packets in $L^2(\mathbb{R}^n)$ defined by $\psi(x) = (\sqrt{\lambda})^{-n} \psi(\frac{x}{\lambda})$ (in the lowest dimensions $n = 1, 2, 3$) with $|\lambda| < 1$ could be achieved as a unitary evolution operation, it would imply that no fundamental limits exist to the possibility of shrinking the particle in an arbitrarily small interval (surface, or volume). Some authors believe that such localization must fail at extremely small scale "below Planck distance". However, in many fundamental discussions, the "Planck distance" is used as a magic spell which permits one to formulate almost any hypothesis almost free of any consistency conditions. Yet, if one believes that QM is a linear theory, then even the most "concentrated" wave packets are just the linear combinations of the extended ones. So, the impossibility of an arbitrarily exact localization of microparticles (beyond or below some special limits) can hardly be defended. The theories of non commutative geometry in which the space coordinates fulfill $[x, y] = \sigma \neq 0$ as a fundamental identity, could neither be constructed; it is enough to note that than the simultaneous squeezing transformation $x \rightarrow v x$ and $y \rightarrow v y$ where $v \neq 0$ (except of some nonlinear version of QM) would ruin the non-commutative law.

Even more essential consequences would follow from the inverse operations in which the initial wave packet could be amplified. Some time ago, a group of authors studied the properties of the radiation emitted from the *extended sources*, asking whether some properties of such sources can be reconstructed from the emitted radiation [56]. However, in a linear theory, each extended wave packet is a linear combination (superposition) of the localized ones. The question arises, what would happen if some experiments could *fish* in the emitted photon state some components corresponding to the localized parts of the initial source? Would the initial state of the extended source be "reduced" to one of its localized emission points? The idea, though curious, seems unreal, due to an excess of empirical complications (indeed, the authors of [56] worried rather about the momenta than space localization of the radiation source). The situation, however, could be different for the charged Schrödinger particle states on the plane orthogonal to the symmetry axis of the solenoid in case of amplification.

In fact, the possible coordinate expansion in $2D$, *i.e.* on the plane orthogonal to the solenoid axis, even if applicable to a restricted class of wave packets, would lead to some critical questions concerning the measurement theory. Assuming that the unavoidable clumsiness in measuring the amplified particle position does not contradict the more exact reconstruction of the original packet coordinates, this would lead to the reduction problem which seems to appear in an ample class of experiments such as *interaction free measurement* of Elitzur and Vaidman [57] and the *delayed choice* effects of J.A. Wheeler [11]. In fact, the finally measured position $\tilde{\mathbf{q}} = \lambda \mathbf{q}$ would be an observable commuting with the initial one. Does it repeat the scenario of the "non-demolition" measurement [46, 47]? If so, the observer performing an (imperfect) position measurement of an amplified wave packet in the future could obtain a posteriori (in a cheap way) the more exact data about its position in the past. The problem arises, whether it means that the "reduction of the wave packet" affects also the particle state in the past, in a new case of "delayed choice" experiment of J.A. Wheeler [11]? Yet, the localization must cost some energy (*cf.* Wigner, Yanase *et al* [58, 59, 60]). One might suppose that the energy needed to localize the amplified packet was provided by the reserves in the screen grains (or whatever medium in which the particle was finally absorbed). This would require an assumption that the relatively low energy invested in an imprecise particle localization in the future should be pumped into much higher energy needed for more precise localization in the past. This seems impossible – not just due to the causality paradox, but also, due to the energy deficit!

Some fundamental effects of quantum mechanics seem in question, as the "delayed choice", or the wave packet reduction in general: a trouble in the QM understanding, though perhaps, good news for the "quantum tomography" (because it would mean the possibility of arbitrarily precise scanning of the original wave packets without an excessive energy pumped to the past). Whatever the answer, it seems that the low energy phenomena might be as close (if not closer) to the fundamental problems of quantum theory as the high energy physics.

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